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LETTER TO THE EDITOR

Singularity-structure analysis and Hirota's bilinearisation of the Davey–Stewartson equation

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Abstract. The singularity-structure aspects of the Davey–Stewartson equation is investigated and it is shown that the system possesses the generalised Painlevé property in the sense of Weiss, Tabor and Carnevale. Further, the associated Bäcklund transformation is constructed and Hirota's bilinearisation is obtained straightforwardly from the Painlevé analysis.

The singular-point-structure analysis leading to the Painlevé property for ordinary differential equations (Ablowitz *et al* 1980) plays a very useful role in determining the integrability property of non-linear dynamical systems (Ramani *et al* 1982, Lakshmanan and Sahadevan 1985). Weiss *et al* (1983) (WTC) have generalised the Painlevé test for partial differential equations (PDE) and this method is a useful tool for testing the integrability of non-linear PDE (Weiss 1983, 1984a, b, Steeb *et al* 1984, Sahadevan *et al* 1986, Clarkson *et al* 1986). In this letter, we present a singularity-structure analysis of the Davey–Stewartson equation describing the two-dimensional lumps in non-linear dispersive systems (Anker and Freeman 1978, Satsuma and Ablowitz 1979) given by

$$\mathrm{i}A_t - \sigma_1 A_{xx} + A_{yy} = \sigma_2 A|A|^2 + 2\sigma_1\sigma_2 QA \qquad (1a)$$

$$\sigma_1 Q_{xx} + Q_{yy} + (|A|^2)_{xx} = 0 \qquad (1b)$$

where $\sigma_i = \pm 1$, $i = 1, 2$, and show that (1) is free from movable critical singularity manifolds. A PDE is said to possess the Painlevé property if its solution can be expressed as a single-valued expansion about its 'non-characteristic' movable singularity manifold

$$\phi(z_1, z_2, \ldots, z_n) = 0. \qquad (2)$$

Thus, the solution $u = u(z_1, z_2, \ldots, z_n)$ of the PDE is single valued and admits a Laurent series expansion around the singularity manifold $\phi(z_1, z_2, \ldots, z_n) = 0$ as

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j \qquad (3)$$

where $u_j = u_j(z_1, z_2, \ldots, z_n)$, $u_0 \neq 0$, are analytic functions of (z_j) in the neighbourhood of the manifold (2) and α is an integer.

In order to investigate the singularity structure of (1), we make a substitution $A = R + \mathrm{i}S$ where R and S are reals. Then (1) takes the form

$$R_t - \sigma_1 S_{xx} + S_{yy} - \sigma_2 S(R^2 + S^2) - 2\sigma_1\sigma_2 QS = 0 \qquad (4a)$$

$$S_t + \sigma_1 R_{xx} - R_{yy} + \sigma_2 R(R^2 + S^2) + 2\sigma_1\sigma_2 QR = 0 \qquad (4b)$$

$$\sigma_1 Q_{xx} + Q_{yy} + 2(RR_{xx} + SS_{xx} + R_x^2 + S_x^2) = 0. \qquad (4c)$$

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The Painlevé test for the PDE proceeds essentially along four main stages: (i) determination of leading-order behaviours, (ii) identifying the powers at which the arbitrary functions can enter, called resonances, (iii) verifying that a sufficient number of arbitrary functions exist without the introduction of algebraic and logarithmic singularity manifolds, and (iv) establishing the connections with other integrability properties.

Let us assume that the leading orders of the solutions of (4) be

$$R \approx R_0\phi^{\alpha} \qquad S \approx S_0\phi^{\beta} \qquad Q \approx Q_0\phi^{\gamma} \tag{5}$$

where R_0, S_0 and Q_0 are analytic functions of (x, y, t) and α, β, γ are integers to be determined. On substituting (5) into (4) and equating the most singular terms we have

$$\alpha = \beta = -1 \qquad \gamma = -2 \tag{6}$$

and from (4*a*) and (4*b*) we obtain a condition

$$2(\sigma_1\phi_x^2 - \phi_y^2) + \sigma_2(R_0^2 + S_0^2) + 2\sigma_1\sigma_2 Q_0 = 0 \tag{7a}$$

and from (4*c*) we have

$$(\sigma_1\phi_x^2 + \phi_y^2)Q_0 + (R_0^2 + S_0^2)\phi_x^2 = 0. \tag{7b}$$

Thus at this stage, we infer from (7) that one of the functions, say R_0 or S_0, is arbitrary and also we have,

$$(R_0^2 + S_0^2) = (2/\sigma_2)(\sigma_1\phi_x^2 + \phi_y^2) \qquad Q_0 = -(2/\sigma_2)\phi_x^2. \tag{8}$$

For finding the resonances, we make a Laurent series expansion

$$R = R_0\phi^{-1} + \sum_{j=1}^{\infty} R_j\phi^{j-1} \qquad S = S_0\phi^{-1} + \sum_{j=1}^{\infty} S_j\phi^{j-1}$$
$$Q = Q_0\phi^{-2} + \sum_{j=1}^{\infty} Q_j\phi^{j-2} \tag{9}$$

and use it in (4) retaining leading-order terms only.

As a result, we obtain the matrix equation

$$[M][V] = 0 \qquad [V]^{\mathrm{T}} = [R_j, S_j, Q_j] \tag{10a}$$

for the lowest-order coefficients, where

$$[M] = \begin{bmatrix} (j^2-3j)(\sigma_1\phi_x^2 - \phi_y^2) + 2\sigma_2 R_0^2 & 2\sigma_2 R_0 S_0 & 2\sigma_1\sigma_2 R_0 \\ 2\sigma_2 R_0 S_0 & (j^2-3j)(\sigma_1\phi_x^2 - \phi_y^2) + 2\sigma_2 S_0^2 & 2\sigma_1\sigma_2 S_0 \\ 2R_0(j-2)(j-3)\phi_x^2 & 2S_0(j-2)(j-3)\phi_x^2 & (j-2)(j-3)(\sigma_1\phi_x^2 + \phi_y^2) \end{bmatrix}. \tag{10b}$$

The resonances are obtained from the condition det $M = 0$ and using (8) it turns out that

$$j = -1, 0, 2, 3, 3, 4. \tag{11}$$

Obviously, the resonance value at $j = -1$ represents the arbitrariness of the singularity manifold $\phi(x, y, t) = 0$, while the resonance at $j = 0$ is associated with the arbitrariness of the functions R_0 or S_0 as seen in (8).

In order to check the existence of a sufficient number of arbitrary functions at the other resonance values $j=2, 3, 3, 4$, we substitute the series representations (9) in (4). First collecting the coefficients of $(\phi^{-2}, \phi^{-2}, \phi^{-3})$ we obtain the equation

$$\begin{bmatrix} -2(\sigma_1\phi_x^2-\phi_y^2)+2\sigma_2R_0^2 & 2\sigma_2R_0S_0 & 2\sigma_1\sigma_2R_0 \\ 2\sigma_2R_0S_0 & -2(\sigma_1\phi_x^2-\phi_y^2)+2\sigma_2S_0^2 & 2\sigma_1\sigma_2S_0 \\ 4\phi_x^2R_0 & 4\phi_x^2S_0 & 2(\sigma_1\phi_x^2+\phi_y^2) \end{bmatrix}\begin{bmatrix} R_1 \\ S_1 \\ Q_1 \end{bmatrix}=[X] \tag{12a}$$

where the column matrix $[X]$ is given by

$$[X]=\begin{bmatrix} S_0\phi_t+2(\sigma_1R_{0x}\phi_x-R_{0y}\phi_y)+R_0(\sigma_1\phi_{xx}-\phi_{yy}) \\ -R_0\phi_t+2(\sigma_1S_{0x}\phi_x-S_{0y}\phi_y)+S_0(\sigma_1\phi_{xx}-\phi_{yy}) \\ -(4/\sigma_2)(\phi_x^2\phi_{yy}-\phi_y^2\phi_{xx}) \end{bmatrix}. \tag{12b}$$

Solving (12), we find that

$$R_1=-\tfrac{1}{2}(\sigma_1\phi_x^2-\phi_y^2)^{-1}[S_0\phi_t-R_0(\sigma_1\phi_{xx}-\phi_{yy})+2(\sigma_1R_{0x}\phi_x-R_{0y}\phi_y)] \tag{13a}$$

$$S_1=-\tfrac{1}{2}(\sigma_1\phi_x^2-\phi_y^2)^{-1}[-R_0\phi_t-S_0(\sigma_1\phi_{xx}-\phi_{yy})+2(\sigma_1S_{0x}\phi_x-S_{0y}\phi_y)] \tag{13b}$$

$$Q_1=(2/\sigma_2)\phi_{xx} \tag{13c}$$

and also that

$$(R_0R_1+S_0S_1)=-(1/\sigma_2)(\sigma_1\phi_{xx}+\phi_{yy}). \tag{13d}$$

Similarly from the coefficients of $(\phi^{-1}, \phi^{-1}, \phi^{-2})$ we obtain

$$\begin{aligned}[-2(\sigma_1\phi_x^2-\phi_y^2)+2\sigma_2R_0^2]R_2+(2\sigma_2R_0S_0)S_2+(2\sigma_1\sigma_2R_0)Q_2 \\ =-[S_{0t}+(\sigma_1R_{0xx}-R_{0yy})+2\sigma_2R_1(R_0R_1+S_0S_1) \\ +\sigma_2R_0(R_1^2+S_1^2)+2\sigma_1\sigma_2R_1Q_1)]\end{aligned} \tag{14a}$$

$$\begin{aligned}(2\sigma_2R_0S_0)R_2+[-2(\sigma_1\phi_x^2-\phi_y^2)+2\sigma_2S_0^2]S_2+(2\sigma_1\sigma_2S_0)Q_2 \\ =-[-R_{0t}+(\sigma_1S_{0xx}-S_{0yy})+2\sigma_2S_1(R_0R_1+S_0S_1) \\ +\sigma_2S_0(R_1^2+S_1^2)+2\sigma_1\sigma_2S_1Q_1)].\end{aligned} \tag{14b}$$

From (14), it is obvious that one of the functions, say Q_2, is arbitrary which corresponds to the resonance value at $j=2$. Solving (14), we have

$$\begin{aligned}R_2=\{&\tfrac{1}{2}(\sigma_1\phi_x^2-\phi_y^2)^{-1}[S_{0t}+(\sigma_1R_{0xx}-R_{0yy})+2R_1(\sigma_1\phi_{xx}+\phi_{yy})] \\ &-\tfrac{1}{2}\sigma_2R_0(\sigma_1\phi_x^2+3\phi_y^2)^{-1}[2\sigma_1Q_2+(R_1^2+S_1^2)] \\ &-\tfrac{1}{2}\sigma_2R_0(\sigma_1\phi_x^2-\phi_y^2)^{-1}(\sigma_1\phi_x^2+3\phi_y^2)^{-1}[(R_0S_{0t}-S_0R_{0t}) \\ &+R_0(\sigma_1R_{0xx}-R_{0yy})+S_0(\sigma_1S_{0xx}-S_{0yy}) \\ &-(2/\sigma_2)(\sigma_1\phi_{xx}+\phi_{yy})(\sigma_1\phi_{xx}-\phi_{yy})]\}\end{aligned} \tag{15a}$$

$$\begin{aligned}S_2=\{&\tfrac{1}{2}(\sigma_1\phi_x^2-\phi_y^2)^{-1}[-R_{0t}+(\sigma_1S_{0xx}-S_{0yy})+2S_1(\sigma_1\phi_{xx}+\phi_{yy})] \\ &-\tfrac{1}{2}\sigma_2S_0(\sigma_1\phi_x^2+3\phi_y^2)^{-1}[2\sigma_1Q_2+(R_1^2+S_1^2)] \\ &-\tfrac{1}{2}\sigma_2S_0[(\sigma_1\phi_x^2-\phi_y^2)^{-1}(\sigma_1\phi_x^2+3\phi_y^2)^{-1}][(R_0S_{0t}-S_0R_{0t}) \\ &+R_0(\sigma_1R_{0xx}-R_{0yy})+S_0(\sigma_1S_{0xx}-S_{0yy}) \\ &-(2/\sigma_2)(\sigma_1\phi_{xx}+\phi_{yy})(\sigma_1\phi_{xx}-\phi_{yy})]\}.\end{aligned} \tag{15b}$$

Proceeding further to the coefficients $(\phi^0, \phi^0, \phi^{-1})$, we have

$$(R_0R_3+S_0S_3+\sigma_1Q_3)=\{-(1/2\sigma_2R_0)[S_{1t}+S_2\phi_t+(\sigma_1R_{1xx}-R_{1yy})+(\sigma_1\phi_{xx}-\phi_{yy})R_2 +2(\sigma_1R_{2x}\phi_x-R_{2y}\phi_y)+2\sigma_2R_2(R_0R_1+S_0S_1)+2\sigma_2R_1(R_0R_2+S_0S_2) +2\sigma_2R_0(R_1R_2+S_1S_2)+\sigma_2R_1(R_1^2+S_1^2)+2\sigma_1\sigma_2(R_1Q_2+R_2Q_1)]\} \tag{16a}$$

$$(R_0R_3+S_0S_3+\sigma_1Q_3)=\{(1/2\sigma_2S_0)[R_{1t}+R_2\phi_t-(\sigma_1S_{1xx}-S_{1yy})-S_2(\sigma_1\phi_{xx}-\phi_{yy}) -2(\sigma_1S_{2x}\phi_x-S_{2y}\phi_y)-2\sigma_2S_2(S_0S_1+R_0R_1)-2\sigma_2S_1(S_0S_2+R_0R_2) -2\sigma_2S_0(S_1S_2+R_1R_2)-\sigma_2S_1(R_1^2+S_1^2)-2\sigma_1\sigma_2(S_1Q_2+S_2Q_1)]\}. \tag{16b}$$

Again one of the functions, say Q_3, is arbitrary. Since we have a double resonance at $j=3$ we require further that either R_3 or S_3 is arbitrary. The actual demonstration of this fact becomes quite tedious for the general manifold. Instead, we have used the Kruskal ansatz (Tabor and Gibbon 1986) which simplifies the computations. Assuming $\phi(x,y,t)=\{x\pm f(y,t)\}$ and substituting (8), (13) and (15) into the right-hand sides of (16*a*, *b*), we find that they are identical and hence the function R_3 (or S_3) becomes arbitrary. Proceeding further to the coefficients of (ϕ^1, ϕ^1, ϕ^0) we have checked that one of the functions of R_4, S_4 or Q_4 is arbitrary. Thus the general solution $\{R(x,y,t), S(x,y,t), Q(x,y,t)\}$ of (4) admits the required arbitrary functions without the introduction of movable critical manifolds and hence the Painlevé property is satisfied for the Davey-Stewartson equation and hence the system (1) is expected to be integrable. Anker and Freeman (1978) have shown that the system (1) belongs to the class of non-linear evolution equation where IST is applicable and also the integrability has been discussed by Fokas and Papageorgiou (1987).

Now, we wish to construct the associated Bäcklund transformations of the Davey-Stewartson equation. For this purpose, we truncate the series up to the constant-level term, that is $R_j=S_j=0$, $j\geqslant 2$ and $Q_j=0$, $j\geqslant 3$. Thus from (9) we have

$$R=R_0\phi^{-1}+R_1 \qquad S=S_0\phi^{-1}+S_1 \qquad Q=Q_0\phi^{-2}+Q_1\phi^{-1}+Q_2 \tag{17}$$

where (R, R_1), (S, S_1) and (Q, Q_2) satisfy (4) with (R_0, S_0, Q_0) satisfying (7) and the remaining ϕ coefficients equated to zero $(Q_1=2\sigma_2^{-1}\phi_{xx})$.

Without loss of generality, we consider the vacuum solutions $R_1=S_1=Q_2=0$ in the Bäcklund transformation (17). Then we have

$$R=R_0\phi^{-1} \tag{18a}$$

$$S=S_0\phi^{-1} \tag{18b}$$

$$Q=Q_0\phi^{-2}+Q_1\phi^{-1}=(2/\sigma_2)(\log\phi)_{xx}. \tag{18c}$$

Now defining $g=(R_0+\mathrm{i}S_0)$ we have

$$A=(R+\mathrm{i}S)=(R_0+\mathrm{i}S_0)\phi^{-1}=g\phi^{-1} \qquad A^*=g^*\phi^{-1}. \tag{18d}$$

On substituting (18*c*) and (18*d*) in the original equation (1) and making use of the Hirota D operators (Hirota 1974, 1980)

$$D_x^M g\cdot\phi=D_x^{M-1}(g_x\phi-g\phi_x) \tag{19a}$$

$$(\partial^2/\partial x^2)(\log\phi)=(1/2\phi^2)D_x^2\phi\cdot\phi \tag{19b}$$

we obtain the following Hirota bilinear form of the Davey-Stewartson equation (1) straightforwardly as

$$(\mathrm{i}D_t - \sigma_1 D_x^2 + D_y^2 - \sigma_2 Y^2) g \cdot \phi = 0 \qquad (20a)$$

$$(\sigma_1 D_x^2 + D_y^2 - \sigma_2 Y^2) \phi \cdot \phi = -\sigma_2 g \cdot g^* \qquad (20b)$$

where $Y = \text{constant}$.

If we expand g and ϕ as power series (Hiota 1974) and use them in (20) we can construct the N-soliton solution in the usual way. Similar bilinearisation (20) has also been reported by Satusma and Ablowitz (1979).

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